International Refereed Journal of Engineering and Science (IRJES)
ISSN (Online) 2319-183X, (Print) 2319-1821
Volume 2, Issue 2(February 2013), PP.25-29
www.irjes.com

Schwarz-Christoffel Transformation On A Half Plane

Abdualah Ibrahim Sultan
Mathematics Department of Brawijaya University Malang, Indonesia.
Mathematics Department of, Faculty of Science, Khoms, Al-Mergeb University ,
Khoms, Libya.

ABSTRACT :*The Schwarz-Christoffel transformation has been studied by many reseachers due to a huge of its application in solving Laplace's equations and studying fluid flow phenomenas (see [3],[4]). The proof of this theorem can find in (see for example [5]). Their method depend on the tangent of* $T_j = 1 + i0$ *and the rotation of* T_j *from* 0 *to* π *on points* $(x, 0)$ *. In this paper we would like to prove the Shwarz-Christoffeltransformation directly by using the arguments identity of complex number, rotation of it from* 0 *to* 2π *and conformal mapping.*

Keywords *: Complex function, half plane, polygon, analytic function, transformation, Conformal mapping.*

I. INTRODUCTION

The Schwarz-Christoffel transformation known as a map from the upper half-plane to simply connected polygon with or without corners at infinity in the complex planes.The Schwarz-Christoffel transformation has been studied by many reseachers due to a huge of its application for example used in physical applications involving fluid movement, heat conduction and electrostatic pontential, as well asthe studying of Laplace's equations (see [3],[4]) and also from mathematical point of views (see [1],[2],[6]).

In this paper we study about the proof of the Schwarz-Christoffel theorem. The prove of this theorem can found in (see for example [5]). Their method depend on the tangent of $T_j = 1 + i0$ and the rotation of T_j from 0 to π on points $(x, 0)$. In this paper we would like to prove the Shwarz-Christoffel transformation directly by using the arguments identity of complex number, rotation of it from 0 to 2π and conformal mapping.

II. PRELIMINARIES

We need to review some important definitions and theorems which will be used later from complex analysis to understand the Schwarz-Christoffeltheorem.These definitions and theorems below can be found in the refences (see [3], [4]).

Definition 2.1 A complex number is a number of the form $z = a + ib$ where the imaginary unit is defined as $i = \sqrt{-1}$and$a$ is the real part of z, $a = Re(z)$, and b is the imaginary part of z , $b = Im(z)$. The set of all complex number is written by C.

Definition 2.2. Arguments of complex number

Let r and θ be polar coordinates of the point (a, b)that corresponds to a *nonzero* complex number $z = a + ib$. Since $a = \cos\theta$and $b = \sin\theta$, the number $z$can be written in *polar form* as $z = \text{r}\,(\cos\theta + i\sin\theta) = |z|e^{i\theta}$, where $|z|$ is a positive real number called the modulus of z, and θ is real number called the argument of z, and written by $\arg\ z$.The real numberθ represents the angle and measured in radians.

Theorem 2.1. Law of arguments of products .

Let $b, c \in C$. Then

$$\arg bc = \arg\ b + \arg\ c. \tag{1.1}$$

Theorem 2.2. Law of arguments of power

Let $b, c \in C$. Then

$$\arg b^c = c \arg b. \tag{1.2}$$

Definition 2.3.Complex Function. *A* complex function is a function f whose domain and range are subsets of the set C of complex numbers.

Definition 2.4.Conformal mapping. Let $w = f(z)$ be a complex mapping defined in a domain D and let z_0 be a point in D. Then we say that $w = f(z)$ is conformal at z_0 if for every pair of smoothoriented curvesC_1 and C_2 in D intersecting at z_0 the angle between C_1 and C_2 atz_0 is equal tothe angle between the image curves $C'_1$and C'_2 at $f(z_0)$ in both magnitude and sense.

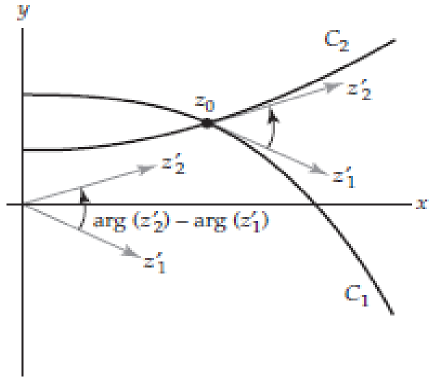

Figure 1.1 The angle betweenC_1 and C_2 (see [4])

In other words a mapping $w = f\ (z)$ preserves both angle and shape but it cannot in general the size. Moreover a mapping that is conformal at every point in a domain D is called conformal in D, the conformal mapping relies on the properties of analytic functio*n*. The angle between any two intersecting arcs in the z-plane is equal to the angle between the images of the arcs in the w-plane under a linear mapping.

Definition 2.5. Analytic Function.A function $f\ (z)$ is said to be analytic in a region R of the complex plane if $f\ (z)$ has a derivative at each point of R. In addition, analytic function is conformal at z_0 if and only if $f^{'}(z)\ \neq 0$.

Definition 2.6.Polygon. A polygon is a plane figure that is bounded by closed path, composed of a finite sequence of straight-line segments (i.e. a closed polygonal *chain* or *circuit*). These segments are called its *edges* or *sides*, and the points where two edges meet are the polygon's *vertices* (singular: vertex) or *corners*.

III. MAIN THEOREM

Theorem (Schwarz-christoffel)

Let P be a polygon in the w-plane with vertices $w_1, w_2, \dots, w_n$ and exterior angles α_k , where $-\pi < \alpha_k < \pi$ .There exists a one-to-one conformal mapping $w = f\ (z)$ from the upper half plane, $Im\ (z) > 0$ onto G that satisfies the boundary conditions in equations $w_k = f\ (x_k)$ for $k = 1,2\ \dots .\ n-1$ and $w_n = f(\infty)$, where $x_1 < x_2 < x_3 < \cdots < x_{n-1} < \infty$. Derivative $f\ '\ (z)$ is

$$f^{'}(z) = A\ (z - x_1))^{\frac{-\alpha_1}{\pi}}\ \ (z - x_2)^{\frac{-\alpha_2}{\pi}} \dots (z - x_{n-1})^{\frac{-\alpha_{n-1}}{\pi}} \tag{1.3}$$

and the function f can be expressed as an indefinite integral

$$f(z) = B + A \int f^{'}(z) = A\ (z - x_1))^{\frac{-\alpha_1}{\pi}}\ \ (z - x_2)^{\frac{-\alpha_2}{\pi}} \dots (z - x_{n-1})^{\frac{-\alpha_{n-1}}{\pi}}\ dz \tag{1.4}$$

where A and B are suitably chosen constants. Two of the points $\{x_k\}$ may be chosen arbitrarily, and the constants A and B determine the size and position of P.

IV. PROVE OF THE THEOREM

A transformation $w = f(z)$ constructed by mapping each point on the x axis in z-plane z on w-plane, with $x_1, x_2, \dots, x_{n-1}, x_n$are the points on the x-axis and $x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n$where the points on the plane z is the domain of the transformation. These points are mapped in to points on the n-side, is $w_j = f(x_j), j = 1, 2, \dots, n$ -1 and $w_n = f(\infty)$in
w-plane. The function $f$chosen so that $\arg f^{'}\ (z)$different constant value

$$f^{'}(z) = A(z - x_1)^{-k_1}(z - x_2)^{-k_2} \dots (z - x_{n-1})^{-k_{n-1}} \tag{1.5}$$

where A is a complex constant and each k_j, with $j = 1, 2, 3, n-1$ are real constants.
From equation (1.5) we taking the absolute value, then we have

$$|f^{'}(z)| = |A(z - x_1)^{-k_1}(z - x_2)^{-k_2} \dots (z - x_{n-1})^{k_{n-1}}|$$

$$=|A\ (z - x_1)^{-k_1}||(z - x_2)^{-k_2}| \dots |(z - x_{n-1})^{-k_{n-1}}|$$

$$=\left|\frac{A}{(z-x_1)^{k_1}}\right| \left|\frac{1}{(z-x_2)^{k_2}}\right| \dots \left|\frac{1}{(z-x_{n-1})^{k_{n-1}}}\right|,$$

by using $(z-x_j)^{-k_j} = |z-x_j|^{-k_j}\exp(-ik_j\theta_j)$, $(-\frac{\pi}{2}<\theta_j<\frac{3\pi}{2})$ where $\theta_j = \arg(z-x_j)$ and j = 1, 2,..., n – 1, then $f'(z)$ is analytic everywhere in the half plane, so

$$|f'(z)| = \left|\frac{Ae^{-ik_1\theta_1}}{(z-x_1)^{k_1}}\right|\left|\frac{e^{-ik_2\theta_2}}{(z-x_2)^{k_2}}\right|\dots\left|\frac{e^{-ik_{n-1}\theta_{n-1}}}{(z-x_{n-1})^{k_{n-1}}}\right|$$

$$= \frac{|A|\left|e^{-ik_1\theta_1}\right|}{|z-x_1|^{k_1}}\frac{\left|e^{-ik_2\theta_2}\right|}{|z-x_2|^{k_2}}\dots\frac{\left|e^{-ik_{n-1}\theta_{n-1}}\right|}{|z-x_{n-1}|^{k_{n-1}}}$$

$$< \frac{A}{|z-x_1|^{k_1}}\frac{1}{|z-x_2|^{k_2}}\dots\frac{1}{|z-x_{n-1}|^{k_{n-1}}} \tag{1.6}$$

By using $\left|e^{-ik_{1j}\theta_j}\right| = 1$ and

$$\frac{2}{|Z|} < |z-x_j| < 2|Z| \text{ and}$$

$$\frac{1}{2|Z|} < \frac{1}{|Z-x_j|} < \frac{2}{|Z|}$$

to equation (1.6) we have that

$$equation\ (1.6) < \frac{2A}{|z|^{k_1}} < \frac{1}{|z|^{k_2}}\dots\frac{1}{|z|^{k_{n-1}}}$$

$= \frac{2A}{|z|^{k_1k_2+\dots+k_{n-1}}}$. Since $k_1+k_2+\dots+k_{n-1}+x_n = 2$,$-1<k_j<1$, $(j = 1.2,\dots,n)$ then

$\frac{2A}{|z|^{k_1k_2+\dots+k_{n-1}}} = \frac{2A}{|z|^{2-k_n}}$. It means, that there exist $M = 2A > 0$ such that

$$|f'(z)| \le \frac{M}{|z|^{2-k_n}}. \tag{1.7}$$

Using equation (1.7) and

$$\lim_{z\to\infty}F(z) = \lim_{z\to\infty}\int_{z_0}^{z} f'(s)ds \tag{1.8}$$

$$\le \lim_{z\to\infty}\left|\int_{z_0}^{z} f'(s)ds\right|$$

$$\le \lim_{z\to\infty}\int_{z_0}^{z} |f'(s)|\,ds$$

$$\le \lim_{z\to\infty}\int_{z_0}^{z}\left|\frac{M}{|z|^{2-k_n}}\right|ds$$

$$= M\lim_{z\to\infty}\int_{z_0}^{z}\frac{1}{|z|^{2-k_n}}ds,$$

since $2-k_n > 1$, then $\int_{z_0}^{z}\frac{1}{|z|^{2-k_n}}ds \to 0$, that means the limit of the integral (1.7) exists as z tend to infinity . So there exists a number w_n such that $f(\infty) =: \lim_{z\to\infty}F(\infty) = w_n, Im\ z \ge 0$. Next we will prove the equation (1.3). After taking the argument to the equation (1.5) we have

$\arg\ f'(z) = \arg A - k_1\arg(z-x_1) - k_2\arg(z-x_2) - \dots - k_{n-1}\arg(z-x_{n-1})$

then by using identity (1.1),(1.2) yields

$$\arg\ f'(z) = \arg\ A + \arg(z-x_1)^{-k_1}\ (z-x_2)^{-k_2} - \dots - \arg(z-x_{n-1})^{-k_{n-1}}$$

$$\arg\ f'(z) = \arg\ A\ (z-x_1)^{-k_1}\ (z-x_2)^{-k_2}\dots(z-x_{n-1})^{-k_{n-1}}$$

$$f'(z) = A\ (z-x_1)^{-k_1}\ (z-x_2)^{-k_2}\dots(z-x_{n-1})^{-k_{n-1}}$$

Let $-k_1 = \frac{-\alpha_1}{x}$ then we have

$$f'(z) = A\ (z-x_1)^{\frac{-\alpha_1}{x}}(z-x_2)^{\frac{-\alpha_2}{x}}\dots(z-x_{n-1})^{\frac{-\alpha_{n-1}}{x}}, \text{ and}$$

$$\int f'(z) = \int A\ (z-x_1)^{\frac{-\alpha_1}{x}}(z-x_2)^{\frac{-\alpha_2}{x}}\dots(z-x_{n-1})^{\frac{-\alpha_{n-1}}{x}}$$

$$f(z)+c = \int A\ (z-x_1)^{\frac{-\alpha_1}{x}}(z-x_2)^{\frac{-\alpha_2}{x}}\dots(z-x_{n-1})^{\frac{-\alpha_{n-1}}{x}}$$

$$f(z) = \int A\ (z-x_1)^{\frac{-\alpha_1}{x}}(z-x_2)^{\frac{-\alpha_2}{x}}\dots(z-x_{n-1})^{\frac{-\alpha_{n-1}}{x}} - c$$

If $-c = B$ then

$$f(z) = B + \int A\ (z-x_1)^{\frac{-\alpha_1}{x}}(z-x_2)^{\frac{-\alpha_2}{x}}\dots(z-x_{n-1})^{\frac{-\alpha_{n-1}}{x}}dz$$

$$= B + \int A\ (z - x_1)^{\frac{-\alpha_1}{x}} (z - x_2)^{\frac{-\alpha_2}{x}} \dots (z - x_{n-1})^{\frac{-\alpha_{n-1}}{x}} dz.$$

Next, since the argument of a product of the complex numbers are equal to the number of arguments of each factor, from equation (1.5) we have

$$\arg f'(z) = \arg A - k_1 \arg(z - x_1) - k_2 \arg(z - x_2) - \cdots - k_{n-1} \arg(z - x_{n-1}). \quad (1.9)$$

The real numbers $x_1, x_2, \dots, x_{n-1},\ x_n$ take on the real axis z-plane. If z is a real number, then the number

$$N_j = z - x_j, \quad (1.10)$$

is positive if z is greater than x_j and negative if$z$ is less than x_j and

$$\arg(z - x_j) = \begin{cases} 0, & \text{if } z > x_j \\ \pi, & \text{if } z < x_j. \end{cases} \quad (1.11)$$

If $z = x$ and $x < x_j, j = 1,2,3, \dots, n-1$ then

$$\arg(z - x_1) = \arg(z - x_2) = \cdots = \arg(z - x_{n-1}) = \pi. \quad (1.12)$$

If $z = x$ and $x_{r-1} < x < x_r$, j = 1, 2, ..., r – 1, r, r + 1, ..., n –1 and let

$$\phi_{r-1} = \arg f'(z) \quad (1.13)$$

then according to equation (1.9) and (1.11) and (1.13)

$$\arg f'(z) = \arg A - k_1 \arg(z - x_1) - k_2 \arg(z - x_2) - \cdots - k_{n-1} \arg(z - x_{n-1})$$
$$\phi_{r-1} = \arg A - k_r \arg(z - x_r) - k_{r+1} \arg(z - x_{r+1}) - \cdots - k_{n-1} \arg(z - x_{n-1})$$
$$= \arg A - k_r\pi - k_{r+1}\pi - k_{r+2}\pi - \cdots - k_{n-1}\pi$$
$$= \arg A - (k_r - k_{r+1} - k_{r+2} - \cdots - k_{n-1})\pi \quad (1.14)$$

and

$$\phi_r = \arg A - k_{r+1} \arg(z - x_{r+1}) - k_{r+2} \arg(z - x_{r+2}) - \cdots - k_{n-1} \arg(z - x_{n-1})$$
$$= \arg A - k_{r+1}\pi - k_{r+2}\pi - k_{r+3}\pi - \cdots - k_{n-1}\pi$$
$$= \arg A - (k_{r+1} - k_{r+2} - k_{r+3} - \cdots - k_{n-1})\pi \quad (1.15)$$

Then

$$\phi_r - \phi_{r-1} = [\arg A - (k_{r+1} - k_{r+2} - k_{r+3} - \cdots - k_{n-1})\pi] - [\arg A - (k_r - k_{r+1} - k_{r+2} - \cdots - k_{n-1})\pi] \quad (1.16)$$

$\phi_r - \phi_{r-1} = k_r\pi$,with $j = 1,2,3, \cdots, r-1, r, \cdots, n-1$.

Such that, if $z = x$ then $x_{j-1} < x < x_j$, $j = 1,2, \dots, n-1$ in the real axis z-plane from the left point x_j to his right, then the vectorτ in w-plane change with angle $k_j\pi$, on the point of the image x_j ,as shown in Figure 1.2. angle $k_j\pi$ is outside the terms of the z-plane at the pointW_j.

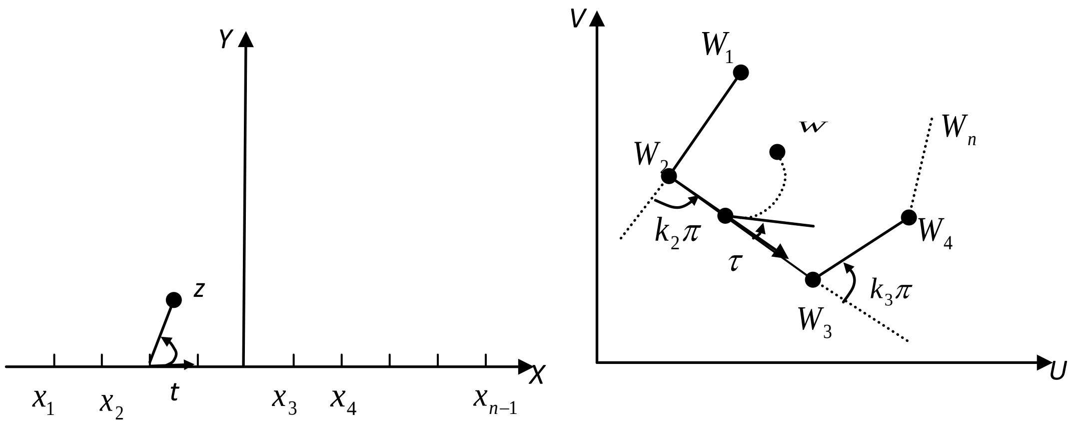

Figure 1.2. Mapping $f(z)$ with $z = x$ and $x_{j-1} < x < x_j$ (see [3])

It is assumed that the sides of the z-plane do not intersect each other and take opposite corners clockwise. Outside corners can be approximated by the angle betweenπ and - π, so that
$-1 < k_j < 1$, If the terms of n-closed, the number of outer corner is 2π.To prove it is seen to have sided closed (n + 1), see Figure 1.3

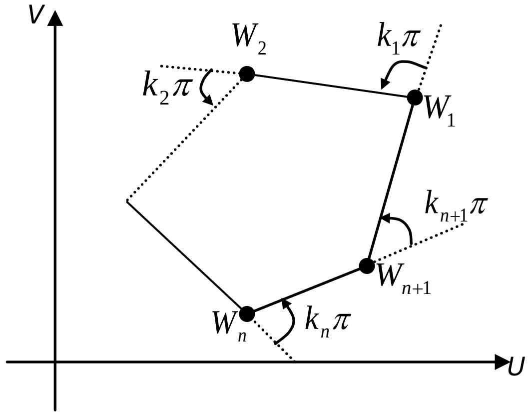

Figure 1.3.Closed n-image has $(n + 1)$ plane (see [3]).

It has known that $k_j\pi$ is the magnitude of the outer corner at the point W_j which is the image of a point x_j.If many terms have side n + 1, then

$$k_2\pi = \phi_2 - \phi_1$$
$$k_3\pi = \phi_3 - \phi_2$$
$$k_4\pi = \phi_4 - \phi_3$$
$$\vdots$$
$$k_n\pi = \phi_n - \phi_{n-1}$$

While the angle $k_1\pi = \phi_1 - \phi_{n+1}$ and $k_{n+1}\pi = \phi_{n+1} + 2\pi - \phi_n$.
In terms of -*n*. $k_{n+1}\pi = 0$ so that $\phi_n = \phi_{n+1} + 2\pi$. Thus, the

$$k_1\pi + k_2\pi + k_3\pi + \cdots + k_n\pi = (\phi_1 - \phi_{n+1}) + (\phi_2 - \phi_1) + (\phi_3 - \phi_2) + \cdots + (\phi_n - \phi_{n+1})$$
$$(k_1 + k_2 + k_3 + \cdots + k_n)\pi = (\phi_n - \phi_{n+1})$$
$$= 2\pi$$
$$k_1 + k_2 + k_3 + \cdots + k_n = 2 \quad (1.17)$$

So it is clear that k_j with $j = 1, 2, 3, \ldots, n$satisfy the condition

$$k_1 + k_2 + k_3 + \cdots + k_n = 2\,, -1 < k_j < 1.$$

End of proof.

V. ACKNOWLEDGEMENTS

I would like to thankProf.Dr.Marjono and Dr. Bagus, for theencouragements.

REFERENCES

[1]. Mark J. Ablowitz Athanassios S.fokas. *Complex variables : Introduction andapplications*, *Second edition*.Cambridge texts in applied mathematics,2003.
[2]. Howell, L.H. *Computation of conformal maps by modified Schwarz-Christoffel transformations*, 1990.
[3]. James Ward Brown and Ruel V.Churchil. *Complex Variables and Applictions, Eighth Edition*, McGraw-Hill, 2009.
[4]. Dennis G.Zill. *Complex Analysis with Applications*, Jones and Bartlett Publisher, 2003.
[5]. Jhon Mathews and Russell Howell. *Complex analysis for mathmatics and engineering sixth edition*,Jones and Bartlett Publisher, 2012.
[6]. Tobin A.Driscoll and Lloyd N. Thefethen. *Schwarz-Christoffel Mapping*, Cambridge Monographs on Applied and Computational Mathematics, 2002.
[7]. Gonzalo Riera , Herna´n Carrasco, and Rube´n Preiss .*The Schwarz-Christoffel conformal mapping for "Polygons" with Infinitely Many Sides*,InternationalJournal of Mathematics and Mathematical, Sciences Volume 2008, Hindawi Publishing Corporation, 2008.